

Different companies use different technical procedures for preparing PEO. WSR-301 PEO is synthesized in the form of structural units, i.e., even in the dry state, this polymer has a complex aggregation of individual macromolecules in groups. This aggregation is evidently reflected in the size of the drag resistance produced in turbulent flow.

#### NOTATION

$g$ , velocity gradient,  $\text{sec}^{-1}$ ;  $[\eta]$ , characteristic viscosity of solution,  $\text{dl/g}$ ;  $M$ , molecular weight of polymer;  $\Delta$ , half-width of scattered-light spectrum,  $\text{Hz}$ ;  $\nu_0$ , frequency of incident radiation,  $\text{Hz}$ ;  $D$ , diffusion coefficient of Brownian particles,  $\text{cm}^2/\text{sec}$ ;  $k$ , Boltzmann constant,  $\text{erg/deg}$ ;  $T$ , absolute temperature,  $^\circ\text{K}$ ;  $\eta$ , viscosity of solution,  $\text{P}$ ;  $\alpha$ , particle radius,  $\mu$ ;  $d = 2kT/6\pi\eta D$ , particle diameter,  $\mu$ ;  $K^2$ , wave number;  $\lambda$ , wavelength of radiation used,  $\mu$ ;  $\theta$ , scattering angle,  $\text{deg}$ ;  $c$ , polymer concentration in solution,  $\text{g/cm}^3$ .

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#### A FINITE-DIFFERENCE METHOD FOR SOLVING INVERSE BOUNDARY-VALUE PROBLEMS OF HEAT CONDUCTION

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A finite-difference search method is described for determining the temperature and heat flux on one boundary of the body if the temperature and heat flux on the other boundary are known. The results of numerical experiments, which show that the method has proved to be efficient, are discussed.

Among the methods at our disposal for solving inverse boundary-value problems of heat conduction (IPHC) fully surveyed in [1-3] an important part is played by the variational methods. The latter are based on the minimization of a functional representing a discrepancy measured by using some norm. To solve IPHC in variational formulation one can use [3-9] the method of least squares, the search methods of the gradient type, or the trial-and-error method.

In the present article the method of finite differences is used to solve boundary IPHC based on the search for a temperature function which satisfies the heat-conduction equation and one of the boundary conditions over the entire region under consideration, the other boundary condition being satisfied in a countable set of points only.

An IPHC is considered for a plate ( $0 \leq x \leq L$ ) described by the following system of equations:

$$c\rho \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial t}{\partial x} \right), \quad (0 < x < L, \tau > 0), \quad (1)$$

$$t(x, 0) = \varphi(x), \quad (2)$$

$$t(0, \tau) = \psi(\tau); \quad (3)$$

$$\frac{\partial t(0, \tau)}{\partial x} = \psi_1(\tau). \quad (4)$$

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It is required to find the temperature function  $t(x, \tau)$  in the region  $0 \leq x \leq L$ ,  $\tau > 0$  as well as the heat flux on the boundary  $x = L$ .

It is noted that problems of heat conduction in a plate for which one of the boundary conditions (3) or (4) is replaced by the temperature at some internal point  $0 < x < L$  are similarly divided into direct and inverse problems.

For the incorrectly formulated IPHC (1)-(2) to become a direct problem of heat conduction it is sufficient that one of the conditions at the boundary  $x = 0$  be replaced, for example, (4), by giving the temperature

$$t(L, \tau) = z(\tau). \quad (5)$$

The solution of (1)-(3), (5) will be the sought solution of (1)-(4) provided a function  $z$  is found such that the condition (4) is satisfied. A finite-difference pattern for finding such a function  $z$  is discussed below on a given set of points so that the condition (4) is satisfied. The same pattern can be used to determine the heat flux  $Q$  on the boundary  $X = L$  instead of the temperature  $t(L, \tau)$ . However, numerical experiments have shown that under almost the same conditions the finding of the temperature of the body boundary is carried out with a much higher accuracy as compared with that of heat flux. It is, therefore, advisable to find  $Q$  after one has calculated  $t(L, \tau)$ .

If an approximation to the function  $z(t)$  is known, then the function  $t(x, t)$  can be found, for example, from an explicit finite-difference pattern which on the grid  $x_i = ih$ ,  $i = 0, 1, \dots, I$ ,  $I = L/h$ ,  $h = \text{const}$ ;  $\tau_n = n\bar{\tau}$ ,  $n = 0, 1, \dots$ , can be represented as

$$u_i^{n+1} = u_i^n \left( 1 - 2 \frac{\lambda l}{c\rho h^2} \right) + \frac{\lambda l}{c\rho h^2} (u_{i+1}^n + u_{i-1}^n) + \frac{l}{c\rho} \frac{\partial \lambda}{\partial t} \left( \frac{u_{i+1}^n - u_{i-1}^n}{2h} \right)^2, \quad (6)$$

$$i = 1, 2, \dots, I-1;$$

$$u_i^0 = \varphi(x_i), u_0^{n+1} = \psi(\tau_{n+1}), u_I^{n+1} = z(\tau_{n+1}). \quad (7)$$

By solving (6)-(7) one can find approximate values of the heat-flux function  $\psi_1^{\bar{\tau}}$  on different temporal layers  $\tau_n$  which correspond to the function  $z(\tau)$ . The difference  $\psi_1(\tau_n) - \psi_1^{\bar{\tau}}$  can be used as a discrepancy signal for subsequent improvement of the function  $z(\tau)$ . In carrying out successive approximations to determine the functions  $u_i^{\bar{\tau}}$  it is more expedient not to evaluate the difference  $\psi_1(\tau_n) - \psi_1^{\bar{\tau}}$  but the temperature difference  $t(h, \tau_n) - u_1^{\bar{\tau}}$  at the internal node point  $x = h$  which is close to the boundary  $x = 0$ . To determine  $t(h, \tau_n)$  one can use either of the difference equations

$$t(h, \tau_n) = t(0, \tau_n) + h \frac{\partial t(0, \tau_n)}{\partial x} + O(h^2) = \psi(\tau_n) + h\psi_1(\tau_n) + O(h^2), \quad (8)$$

$$t(h, \tau_n) = \psi(\tau_n) + h\psi_1(\tau_n) + \frac{h^2}{2} \left[ \frac{c\rho}{\lambda} \frac{\partial \psi(\tau_n)}{\partial \tau} - \frac{1}{\lambda} \frac{\partial \lambda}{\partial t} \psi_1^2(\tau_n) \right] + O(h^3).$$

The function  $z(\tau)$  is found by iterating on steps on the  $\tau$  axis. The step  $\Delta\tau$ , within which the function  $z(\tau)$  is determined, should be chosen by employing the condition that the heat perturbations which arise during the period  $\Delta\tau$  on the boundary  $x = L$  should appear sufficiently distinctly on the boundary  $x = 0$  during the same period of time. For example, if on the boundaries of a uniformly heated plate an instantaneous change of temperature has taken place [10] by a quantity  $\Delta t$ , then the temperature in the middle of the plate will change by  $0.01\Delta t$  for  $Fo = 0.06$ , by  $0.05\Delta t$  for  $Fo = 0.1$ , and by  $0.22\Delta t$  for  $Fo = 0.3$ .

The above condition imposes a constraint on the smallest value of  $\Delta\tau$ . This results in the inadequacy of the linear approximation of the function  $z(\tau)$  in the interval  $\Delta\tau$ . It is assumed, therefore, that the temperature function  $z(\tau) = t(L, \tau)$  is sufficiently smooth in the interval  $\tau_j < \tau < \tau_j + \Delta\tau$ . The function  $z(\tau)$  is expanded into a Taylor series and transformed into

$$z(\tau) = z(\tau_j) + a_0^j(\tau - \tau_j) + a_1^j(\tau - \tau_j) \left( 1 - \frac{\tau - \tau_j}{\Delta\tau} \right) + a_2^j(\tau - \tau_j)^2 \left( 1 - \frac{\tau - \tau_j}{\Delta\tau} \right) + \dots \quad (9)$$

The use of the expansion (9) is convenient in the sense that the value of  $z(\tau_j + \Delta\tau)$  at the right end of the interval only depends on the parameter  $a_0^j$ . It is advisable to choose the interval  $\Delta\tau$  as a multiple of  $i(S + 1)$ , where  $S$  is the subscript of the coefficient  $a_S^j$  in the

last term retained in the series (9). In the evaluation of the coefficients  $\alpha_s^j$ ,  $s = 0, 1, \dots, S$ , of the series (9) a specific point  $\tau = \tau_j + p\delta\tau$ ,  $p = 1, 2, \dots, S + 1$ ,  $\delta\tau = \Delta\tau/(S + 1)$  of the interval  $[\tau_j, \tau_j + \Delta\tau]$  is associated with each of them. For example, with the coefficient  $\alpha_0^j$ , which to considerable degree determines the function  $z(\tau)$  in the interval  $[\tau_j, \tau_j + \Delta\tau]$ , there is associated a point  $\tau = \tau_j + \Delta\tau$ . Each coefficient  $\alpha_s^j$  is responsible for the point associated with it to have the values  $t(h, \tau_n)$  and  $u_n^j$  virtually equal, that is, for

$$t(h, \tau_n) - u_n^j < \varepsilon, \quad (10)$$

where  $\varepsilon$  is a small value, which in our numerical experiments has usually been taken as equal to  $10^{-9}$ .

Let us now assume that for some time interval  $[\tau_{j-1}, \tau_{j-1} + \Delta\tau]$  the function  $z(\tau)$  and the coefficients  $\alpha_s^{j-1}$ ,  $s = 0, 1, \dots, S$ , have already been determined and that it is required now to find the function  $z(\tau)$  in the next interval  $[\tau_j, \tau_j + \Delta\tau]$ . The difference  $\tau_j - \tau_{j-1}$  is chosen as either equal to  $\delta\tau$  or to its multiple, since this ensures that the condition (10) is satisfied at the left end  $\tau_j$  of the interval.

The coefficients  $\alpha_s^j$ ,  $s = 0, 1, \dots, S$ , are found by successive approximations over the cycles. Each cycle begins with the first approximation  $\alpha_{s(1)}^j$  and terminates with the approximation  $\alpha_{s(k)}^j$  when the condition (10) is satisfied for the point associated with this coefficient. The last value  $\alpha_{s(k)}^j$  of the preceding cycle is the first approximation  $\alpha_{s(1)}^j$  for the subsequent cycle. Any subsequent approximation for the coefficient  $\alpha_s^j$  is carried out only if the condition (10) has been satisfied at the points of the interval  $[\tau_j, \tau_j + \Delta\tau]$  which are associated with the coefficients  $\alpha_0^j, \alpha_1^j, \dots, \alpha_{s-1}^j$ . Thus, each consecutive approximation of the coefficient  $\alpha_s^j$  indicates the start of another cycle in the evaluation of the coefficients  $\alpha_0^j, \alpha_1^j, \dots, \alpha_{s-1}^j$ . If  $\alpha_{s(k)}^j$  is the value of the coefficient  $\alpha_s^j$  in the  $k$ -th approximation, then to determine  $\alpha_{s(k+1)}^j$  one uses the formula

$$\alpha_{s(k+1)}^j = \alpha_{s(k)}^j + [t(h, \tau^s) - u_{1(k)}^s] \frac{1}{w_{s(k)}^j}, \quad (11)$$

where  $\tau^s$  is the point of the interval  $[\tau_j, \tau_j + \Delta\tau]$  associated with the coefficient  $\alpha_s^j$ ;  $u_{1(k)}^s$  is the value of the grid function  $u_1^s$  on the layer  $\tau^s$  computed directly prior to the  $(k + 1)$ -th approximation of the coefficient  $\alpha_s^j$ ;  $w_{s(k)}^j$  is the absolute value of the rate of change of the grid function  $u_1^s$  with respect to the parameter  $\alpha_s^j$  for the given cycle determined after the second approximation and remaining unchanged in the subsequent approximations:

$$w_{s(2)}^j = \left| \frac{u_{1(2)}^s - u_{1(1)}^s}{\alpha_{s(2)}^j - \alpha_{s(1)}^j} \right|. \quad (12)$$

For the second approximation the rate  $w_{s(1)}^j$  was adopted as equal to the rate in the preceding cycle. By a suitable adoption of the function  $u_1^s$  and by using relations similar to (8), one can determine the heat flux on the boundary  $x = L$ .

Numerical experiments have shown that if one retains only the first three terms in the series (9), a sufficiently high accuracy is ensured for the solution of the problem (1)-(4), provided the temperature does not change too rapidly at the boundary  $x = L$ . Then the coefficients  $\alpha_0^j$  and  $\alpha_1^j$  are associated with the points  $\tau_j + \Delta\tau$  and  $\tau_j + (\Delta\tau/2)$  of the interval  $[\tau_j, \tau_j + \Delta\tau]$ . If the condition (10) is satisfied for the node points  $\tau_j + \Delta\tau$  and  $\tau_j + (\Delta\tau/2)$ , one proceeds to the interval  $[\tau_{j+1}, \tau_{j+1} + \Delta\tau]$ , where  $\tau_{j+1} = \tau_j + (\Delta\tau/2)$ . The computations have shown that one usually needs 4-5 iterations on each time interval to determine the coefficients  $\alpha_0^j$  and  $\alpha_1^j$ . To illustrate the investigated method, the results are given below of a numerical experiment which consisted of the following stages:

1. The solution of the direct (not inverse) problem (1)-(3), (5) to determine the functions  $\psi_1(\tau)$  and  $t(h, \tau)$  for the following initial data:

$$\begin{aligned} t(x, 0) = 1; \quad t(0, \tau) = 1 + b_1\tau; \quad t(L, \tau) = \exp(-b_2\tau) \times \\ \times \sin b_3\tau; \quad h = \frac{L}{10} = 0,1; \quad c\rho = 1; \quad \lambda = 1 + b_4t + b_5t^2; \end{aligned}$$

$$b_m = \text{const}, \quad m = 1, 2, \dots, 8; \quad b_8 = 1 \div 15.$$

2. The perturbation of the functions  $\psi(\tau)$ ,  $\psi_1(\tau)$ , and  $\varphi(x)$  which describe the heat-exchange conditions on the boundaries  $x = 0$  as well as the temperature field at the initial

TABLE 1. Comparison with Exact Solution

$\tau$	$t(L, \tau)$	Values of $t(L, \tau)$ obtained by solving the IPHC			
		$\xi=0; b=0$	$\xi=0.05; b=0$	$\xi=0.1; b=0$	$\xi=0.05$
0,15	1,413	1,414	1,425	1,4364	1,519
0,45	1,839	1,849	1,879	1,908	1,973
0,6	1,797	1,807	1,720	1,632	1,745
0,9	1,316	1,319	1,173	1,027	1,181
1,2	0,7033	0,7070	0,5855	0,4637	0,5628
1,5	0,4071	0,4111	0,3107	0,2856	0,256
1,8	0,5759	0,5761	0,3986	0,2987	0,3254
2,1	1,008	1,007	0,9296	0,8516	0,8579
2,4	1,356	1,358	1,271	1,183	1,221
2,7	1,394	1,396	1,359	1,322	1,344
3	1,151	1,152	1,108	1,065	1,130
3,3	0,8476	0,8487	0,816	0,7843	0,8642

time. To perturb the function  $\psi_1(\tau)$  a pseudo-random-number generator was employed; the numbers had a normal distribution with a mean relative error  $\xi = \Delta\psi/\psi_1 = 0-0.5$ . The perturbation of the temperature  $\psi(\tau)$  was carried out by adding to the function  $\psi$  a sign-changing function  $\Delta t = \psi b \sin b_6 \tau$ ,  $b = 0-0.5$ ,  $b_6 = 0-15$ . The function  $\varphi(x)$  was perturbed by adding the function  $\Delta\varphi = \varphi(x)b_7 \sin b_8 x$ ,  $b_7 = 0-0.15$ ,  $b_8 = 0-100$  to  $\varphi(x)$ .

3. The solving of the IPHC for perturbed data. To solve numerically an IPHC in the time interval  $0 < \tau < 5$  about 6 min of machine time is needed on the BESM-4 electronic computer. If the same grid is used when solving an inverse problem with unperturbed initial data, the error is virtually the same as for the direct problem.

In Table 1 the results of a numerical experiment with exact or perturbed initial data for  $b_1 = 0$ ;  $1/b_2 = b_3 = 3$ ;  $b_4 = b_5 = 0.1$ ;  $b_6 = 4$ ;  $\Delta\tau = 0.3$ ;  $b_7 = b_8 = 0$  are given. It can be seen from the table that if the exact input data are given, the difference between the exact temperature values and those found by solving IPHC does not exceed 1%. Inexact initial data result in greater errors in the grid functions  $u_L^n$ . However, this growth of error is obviously not related to the increase in the computation error; this is confirmed indirectly by the calculations carried out for different rules used for modifying the functions  $t(L, \tau)$  and  $\lambda = \lambda(t)$ . By modifying the error of the initial data the mean error in determining the function  $t(L, \tau)$  via the solution of IPHC changes roughly proportionally. If the mean relative error in finding  $t(L, \tau)$  related to the initial temperature  $t(0, 0)$  is  $\xi_L$ , then in solving the IPHC with the time interval  $\Delta\tau = 0.3$  the ratios of the errors  $\xi_L/\xi$  and  $\xi_L/b$  are approximately 1.3 and 0.85. For  $\Delta\tau = 0.6$  the ratios are  $\xi_L/\xi \approx 0.8$  and  $\xi_L/b \approx 0.9$ .

An error in the given initial starting data for  $\varphi(x)$  has a pronounced effect on the temperature function only for relatively small values of  $Fo < 0.5$ , the error decreasing with the growth of  $b_8$ . The varying of  $b_8$  between 1 and 100 when  $b_7 = 0.1$ ,  $\xi = 0$ ,  $b = 0$  results in the reduction of the maximal error in the determination of the temperature  $t(L, \tau)$  for  $\tau = \Delta\tau/2$  from 11% to 6%.

The above results demonstrate that the described finite-difference pattern for solving IPHC enables one to find the unknown boundary conditions with an error which remains close to the error in the initial data.

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